Cold relativistic helically symmetric steady flows

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A general theory of nontenuous cold relativistic helically symmetric steady flows is developed. Both self-fields and radial effects are included. The fluid and the Maxwell equations are reduced to a system of equations for three scalar functions: a stream function, a flux function, and an electrostatic potential. These equations are expanded under the assumption of small perpendicular momenta and a simplified set of ordinary differential equations is obtained. Two cases are studied in detail. The first case is of a system externally driven by a magnetic wiggler. The second is the self-excited system, with no externally applied wiggler. The equilibria described here are of a practical importance for free-electron lasers which employ high-density thick beams.

I. INTRODUCTION

In recent years there has been a considerable interest in helically symmetric relativistic electron flows, mainly for free-electron laser applications.[1] Helical steady flows were first studied by Friedland[2] using the paraxial approximation. Later analysis by Diament[3] and Freund *et al.*[4] included the radial dependence of the external helical magnetic field but considered a tenuous (i.e., very low-density) beam limit with no self-fields of the beam in the equilibrium. Our purpose is twofold. First we present a cold fluid description of a relativistic non-neutral steady flow which is helically symmetric. Second, we approximate this general description in order to obtain a simplified picture of some important classes of free-electron laser equilibria.

The general theory follows the pattern for steady two-dimensional flows in fluid dynamics and in magnetofluid dynamics, cf. Grad[5] and Weitzner.[6] We introduce a stream function, a flux function, and an electrostatic potential to represent the current, the magnetic field, and the electric field, respectively. The continuity equation for the density, the momentum equation for the momentum vector, and the Maxwell equations for the electric and the magnetic field vector are replaced by equations for the three scalar functions. We discuss the type of these equations and what additional data must be specified to characterize a solution.

In the second part of the paper we present a simplified picture which approximates some free-electron experiments. Previous theories have neglected the self-fields of the beam and solve for the particle trajectories in a given external magnetic field. In our approach we assume that the self-fields of the beam are comparable in magnitude to the external helical magnetic field, a situation typical of high-density beam experiments. We also assume that the perpendicular momentum of the fluid is small relative to the parallel momentum and that their ratio is a small parameter in which we expand the system of equations. To lowest order we have a cylindrical beam moving in a uniform magnetic field with zero perpendicular momentum. The helical effects appear in a higher order in which both the perpendicular momentum and the self-fields of the beam are present. In the process of the expansion of the equations, the system of partial differential equations in two independent variables reduces to a system of ordinary differential equations whose independent variable is the radial coordinate. In the tenuous beam limit, analytic expressions are derived for the momenta, which agree with previous results applicable for low-density beams and near the beam axis.

We first examine the case in which an external helical magnetic field is present. One contribution of this paper is the inclusion of the self-fields of the beam in the equilibria in a consistent manner. However, even in the tenuous beam limit, our results are different from previous ones. Freund *et al.*[4] have developed a special solution in which the beam is a filament in space and all the particle trajectories are axicentered helices. Although particles which enter the interaction region on axis may follow such a centered helix, particles which are initially off axis will probably not do so. The question of steady state orbits of a thick beam particle has recently been addressed by Fajans *et al.*[7] in a semiempirical way. The present study contains a consistent approximate solution for the flow of a thick beam in which the particle trajectories are not axicentered and the beam in lowest order is a cylinder and not a filament. Our steady solution is easily extended to include the self-fields of the beam, while a filament-type solution is not likely to exist in the presence of self-fields.

After studying the steady flow in which an external helical magnetic field ("wiggler") is present, we turn to investigate steady helical flows in the absence of an external driving wiggler. Such beams in the thin, tenuous limit were found to be very unstable at high frequencies and they are the basis of the wiggler-free, free-electron lasers.[8] We study in detail these helical flows when the perpendicular momenta of the beam caused by the self-fields are comparable to the momenta associated with the helical dependence. In order to show the existence of such helical solutions we must examine our system of equations to even higher order than in the case when wiggler fields are present. In higher order the fields generated by the helical motion exert forces that balance each other. This rather involved study demonstrates the possibility of having self-excited, high-density thick helical beams.

In both the case of an externally driven system (with the wiggler) and the case of self-excited system (wiggler-free system), we have considered a general magnetic multipole. The dipole-type flow which is commonly used for free-elec-

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tron lasers and the recently suggested quadrupole wiggler flow[9] are special cases of our analysis.

In Sec. II we present the general formalism, and in Sec. III we start the derivation of the formal expansion of the equations. Next we treat the system driven by an external wiggler in Sec. IV, and the self-excited wiggler-free system in Sec. V. We conclude in Sec. VI.

II. GENERAL FORMALISM

We consider a relativistic but cold-electron gas described by reduced momentum $\tilde{\mathbf{u}}$. The electromagnetic field in Gaussian units are $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{B}}$. The equations that characterize the gas are

conservation of mass

$$\nabla\cdot\tilde{n}\tilde{\mathbf{u}} = 0; \tag{1}$$

conservation of momentum

$$m\tilde{n}(\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}} = e\tilde{n}(\gamma\widetilde{\mathbf{E}} + \tilde{\mathbf{u}}\times\widetilde{\mathbf{B}}/c); \tag{2}$$

and Maxwell's equations

$$\nabla\cdot\widetilde{\mathbf{E}} = 4\pi en\gamma, \tag{3a}$$

$$\nabla\times\widetilde{\mathbf{B}} = 4\pi e\tilde{n}\tilde{\mathbf{u}}, \tag{3b}$$

$$\nabla\times\widetilde{\mathbf{E}} = 0, \tag{3c}$$

where

$$\gamma^2 = 1 + \tilde{\mathbf{u}}\cdot\tilde{\mathbf{u}}/c^2. \tag{4}$$

It is convenient to work with mostly nondimensional variables, and to this end we define

$$\widetilde{\mathbf{E}} = (mc^2/|e|)\mathbf{E}, \quad \widetilde{\mathbf{B}} = (mc^2/|e|)\mathbf{B},$$

$$\tilde{\mathbf{u}} = c\mathbf{u}, \quad \tilde{n} = [mc^2/(4\pi e^2)]n.$$

We recognize that we are concerned with an electron gas so that $|e| = -e$ and we introduce the electrostatic potential such that

$$\mathbf{E} = -\nabla\Phi, \tag{5}$$

and our system becomes

$$\nabla\cdot n\mathbf{u} = 0, \tag{1'}$$

$$(\tilde{\mathbf{u}}\cdot\nabla)\mathbf{u} = \gamma\nabla\Phi - \mathbf{u}\times\mathbf{B}, \tag{2'}$$

$$\Delta\Phi = n\gamma, \tag{3a'}$$

$$\nabla\times\mathbf{B} = -n\mathbf{u}, \tag{3b'}$$

where $\gamma^2 = 1 + \mathbf{u}\cdot\mathbf{u}$. It is convenient to refer to the nondimensional variables n, $\mathbf{u}$, Φ, and $\mathbf{B}$ as the number density, momentum, electrostatic potential, and magnetic field, respectively, although the names are not quite correct.

We limit ourselves to solutions which are helically symmetric. Such solutions are the basic elements of free-electron lasers, and a considerable effort has been given to their study.[2–4,7] A steady flow is said to be helically symmetric, if, when the system is described in cylindrical coordinates r, θ, and z and expressed in terms of vectors in the $\hat{r}$, $\hat{\theta}$, and $\hat{z}$ directions, all quantities are functions of r and

$$\phi \equiv \theta - kz \tag{6}$$

only. The parameter k is the helical wavenumber and specifies the basic periodicity length in z as $2\pi/k$. Helical symmetry enables us to satisfy (1′) and the condition $\nabla\cdot\mathbf{B} = 0$ exactly by the introduction of a stream function $\chi(r,\phi)$ and a flux function $\psi(r,\phi)$ such that

$$nu = -\chi_{,\phi}/r, \tag{7a}$$

$$n(v - krw) = \chi_{,r} \tag{7b}$$

and

$$B_r = -\psi_{,\phi}/r, \tag{8a}$$

$$B_\theta - krB_z = \psi_{,r}, \tag{8b}$$

where

$$\mathbf{u} = u\hat{r} + v\hat{\theta} + w\hat{z}. \tag{9}$$

We employ the common notation for derivatives in which $F_{,x}$ represents the derivative of F with respect to x. Trivial consequences of (7) and (8) are that $n\mathbf{u}\cdot\nabla\chi = 0$ and $\mathbf{B}\cdot\nabla\psi = 0$, which justify the names, respectively, of a stream function and a flux function.

From the $\hat{r}$ and $\hat{\theta} - kr\hat{z}$ components of (3b′) we infer that

$$\chi = B_z + krB_\theta, \tag{10}$$

so that

$$B_\theta = (kr\chi + \psi_{,r})/(1 + k^2r^2) \tag{11a}$$

and

$$B_z = (\chi - kr\psi_{,r})/(1 + k^2r^2). \tag{11b}$$

The remaining information in the system (3b′) is a Grad–Shafranov equation for the flux function ψ

$$\Delta^*\psi = -nw - kr\chi_{,r}/(1 + k^2r^2) - 2k\chi/(1 + k^2r^2)^2, \tag{12}$$

where

$$\Delta^*\psi \equiv [r\psi_{,r}/(1 + k^2r^2)]_{,r}/r + \psi_{,\phi\phi}/r^2. \tag{13}$$

At this point (8a), (11), and (12) replace (3b′). Thus, the Maxwell equation for $\mathbf{B}$ is reduced to (12) with the representations for the fields (8a) and (11).

We now turn to the remaining equation, conservation of momentum (2′). If we dot $\mathbf{u}$ into (2′) we readily find the relativistic form of the Bernoulli law, in this case conservation of energy $(n\mathbf{u}\cdot\nabla)(\gamma - \Phi) = 0$, for which the general solution is

$$\gamma = \Phi + E(\chi), \tag{14}$$

where $E(\chi)$ is an arbitrary function of the stream function χ. When we take the $kr\hat{\theta} + \hat{z}$ component of (2′) we find easily $n\mathbf{u}\cdot\nabla(krv + w + \psi) = 0$, so that

$$krv + w + \psi = F(\chi), \tag{15}$$

where $F(\chi)$ is another arbitrary function of χ. In (14) and (15) we have integrated two of the three components (2′). We shall examine the remaining component shortly, but first we extract further information from (14) and (15). We may solve (7b) and (15) for the momenta v and w to find

$$nv = [\chi_{,r} + krn(F - \psi)]/(1 + k^2r^2), \tag{16a}$$

$$nw = [n(F - \psi) - kr\chi_{,r}]/(1 + k^2r^2). \tag{16b}$$

One consequence of (16b) is that the Grad–Shafranov equation reduces to the simpler form

$$\Delta^*\psi = n[\psi - F(\chi)]/(1 + k^2r^2) - 2k\chi/(1 + k^2r^2)^2. \tag{12'}$$

Next, from (7a), (14), (16), and the nondimensional form of (4) we may solve for n^2 in terms of χ, ψ, Φ, $E(\chi)$, and $F(\chi)$ to find

$$n^2 = |\nabla\chi|^2/\{(1+k^2r^2)[(E+\Phi)^2-1]-(F-\psi)^2\}, \quad (17)$$

where

$$|\nabla\chi|^2 = \chi_{,r}^2 + \chi_{,\phi}^2(1+k^2r^2)/r^2. \quad (18)$$

In helical symmetry we may finally rewrite Poisson's equation (30′) as

$$(r\Phi_{,r})_{,r}/r + (\Phi_{,\phi\phi})(1+k^2r^2)/r^2 = n[\Phi + E(\chi)], \quad (3a'')$$

with n given by (17).

Finally we turn to the remaining information in the equations of conservation of momentum. It is simplest to dot the vector $\nabla\chi$ into (2′) and in view of the identity

$$0 = (n\mathbf{u}\cdot\nabla)(\mathbf{u}\cdot\nabla\chi) = \mathbf{u}\cdot(n\mathbf{u}\cdot\nabla)\nabla\chi + \nabla\chi\cdot(n\mathbf{u}\cdot\nabla)\mathbf{u} \quad (19)$$

we readily find

$$L\chi = -\frac{\chi_{,r}}{r(1+k^2r^2)}\left(\chi_{,\phi}^2 + \frac{k^2r^4n^2(\psi-F)^2}{(1+k^2r^2)}\right) - n^2[\Phi+E(\chi)]r^2\nabla\chi\cdot\nabla\Phi + \frac{n^2r^2(\psi-F)}{(1+k^2r^2)}\nabla\chi\cdot\nabla\psi + \left[\frac{nr^2}{(1+k^2r^2)}\left(\chi + \frac{2k(\psi-F)}{(1+k^2r^2)}\right) - \frac{r\chi_{,r}}{(1+k^2r^2)^2}\right]|\nabla\chi|^2, \quad (20)$$

where

$$L\chi \equiv \chi_{,r}^2\chi_{,\phi\phi} + \chi_{,\phi}^2\chi_{,rr} - 2\chi_{,\phi}\chi_{,r}\chi_{,r\phi}. \quad (21)$$

Thus, the steady flow is characterized by the electrostatic potential Φ, the (magnetic) flux function ψ, and the stream function χ. These functions satisfy the three second order differential equations (3a″), (12′), and (20), which contain two arbitrary functions $E(\chi)$ and $F(\chi)$, where n is given by (17). The electromagnetic fields are given in terms of the generalized potentials by (5), (8a), and (11), while the particle momenta are given by (7a) and (16), with n given by (17).

Of the three differential equations (3a″), (12′), and (20), clearly (3a″) and (12′) are elliptic, so that consistent with physical intuition would expect to provide boundary data for Φ and ψ on some outer boundary. However (20) raises more substantial problems. It is an easy calculation to show that the characteristics of (20), given by the characteristics of (21), are the curves $\chi(r,\phi) = \text{const}$ counted twice. Thus, this equation is a degenerate hyperbolic equation. Two distinct situations are possible and they correspond to different types of data to be imposed. If one solves (20) in a domain in which each characteristic enters the domain at a boundary point and then leaves the domain at another boundary point then one would expect to give two pieces of data on each characteristic, possibly χ and $\partial\chi/\partial n$ at one end of each characteristic, possibly χ or $\partial\chi/\partial n$ at both ends of each characteristic. Such a problem would be typical of supersonic fluid flow, for instance. This type of problem would fall more or less within the conventional view of solution of differential equations.

In many problems of interest, for instance in beam flow problems, the flow is expected to be fully contained within some domain in the (r,ϕ) plane and correspondingly the streamlines $\chi = \text{const}$ are also to be contained in some domain. In this case we expect each streamline to close on itself and it is not at all clear what is correct data for (20). We require that χ be periodic on the streamline; thus we expect χ and the tangential derivative on a streamline to be periodic. Two "jump" conditions are usually counted as one boundary condition, thus we may expect that on each streamline we might give only one piece of data. However, this problem is sufficiently nonstandard, that we cannot be sure that this counting of boundary conditions is correct. The question of how to deal with (20) is quite similar to the problem of existence of nonsymmetric toroidal magnetohydrodynamic equilibria, where again one has a system with one real characteristic counted twice. Our problem is simpler, as we show in the next section, since the resonances are isolated, rather than dense. We are forced to leave these questions unresolved, although we comment further in the next section.

III. APPROXIMATION OF THE EQUATIONS

The solution of the equations described in the previous section is dependent on the resolution of many nontrivial mathematical questions and in any case would require a substantial numerical effort. For some applications, however, it is possible to find approximate solutions. These solutions should also help us select correct numerical algorithms for the full system. If the density of the beam is low, one may neglect the self-fields and solve the equation of motion in given external magnetic fields only. This approach has been widely used in free-electron lasers studies (cf. the references cited in the Introduction). Axicentered trajectories, which are filaments in space, are the basis of some steady flows.[4] However the assumption of an axicentered filament without self-fields is not valid for high-density thick electron beams.

We therefore look for an approximate solution in which the influence of the self-fields is included and the beam is not a filament in space but approximately a cylinder. Since the perpendicular momentum of the particle in free-electron lasers is usually much smaller than the parallel momentum, we choose the mean ratio of the momenta as a small parameter in which the equations are expanded.

We are interested in a flow with a moderately low density, and which is approximately a particle beam with flow momentum in the z direction, but which is a function of r. We also assume that the density is low enough that the magnetic field is only mildly perturbed from a constant field in the z direction. To represent this situation we construct a formal perturbation expansion of the system (3a″), (12′), and (20) in a small parameter ϵ. This parameter is a measure of the ratio of the perpendicular particle momentum to the momentum parallel to the z axis, the magnitude of the number density, and the magnitude of the helical effects. We expand all the dependent variables in formal power series in this parameter in the form

$$g(r,\phi) = \sum_{n=0}^{\infty} \epsilon^n g_n(r,\phi).$$

Since in lowest order we want a constant magnetic field in the z direction, we take in lowest order

$$\chi = \chi_0, \tag{22a}$$

$$\psi = \psi_0(r) \equiv -\tfrac{1}{2}kr^2\chi_0, \tag{22b}$$

$$\Phi = \Phi_0 \equiv 0. \tag{22c}$$

It is trivial to conclude that the system of differential equations is identically satisfied and

$$\mathbf{B}_0 = \chi_0\hat{z}, \quad \mathbf{E}_0 = 0. \tag{23}$$

We do not yet know $\mathbf{u}$ in zeroth order since n is first order and thus $n\mathbf{u}$, given by (7a) and (16), is also first order. We shall have to carry the expansion at least partially through second order before we can exhibit any nontrivial helical dependence in $\mathbf{u}$. We assume a particular form for the unknown functions $E(\chi)$ and $F(\chi)$. Since we may allow $E(\chi)$ and $F(\chi)$ to be explicit functions of ϵ we take

$$E(\chi) = E((\chi - \chi_0)/\epsilon,\epsilon) = E(\chi_1 + \epsilon\chi_2 + \epsilon^2\chi_3 + \cdots,\epsilon), \tag{24a}$$

$$F(\chi) = F(\chi_1 + \epsilon\chi_2 + \epsilon^2\chi_3 + \cdots,\epsilon). \tag{24b}$$

We now examine the system in first order, and consistent with our hypothesis that the flow should be almost entirely in the z direction and only r dependent we assume that through first order

$$\chi = \chi_0 + \epsilon\chi_1(r), \tag{25a}$$

$$\psi = -\tfrac{1}{2}kr^2\chi_0 + \epsilon\psi_1(r,\phi), \tag{25b}$$

$$\Phi = \epsilon\Phi_1(r). \tag{25c}$$

We find from (17)

$$n_1^2 = (\chi_{1,r})^2/\{(1+k^2r^2)[E_0^2(\chi_1(r)) - 1] - [F_0(\chi_1(r)) - \psi_0(r)]^2\} \tag{26}$$

and thus

$$\Delta^*\psi_1 = n_1[\psi_0 - F_0(\chi_1(r))]/(1+k^2r^2) - 2k\chi_1/(1+k^2r^2)^2, \tag{27}$$

$$\Delta\Phi_1 = n_1E_0(\chi_1(r)). \tag{28}$$

If we examine (20), then the left-hand side is $O(\epsilon^4)$, thus all lower-order terms on the right-hand side must vanish. In particular if we select the $O(\epsilon^3)$ terms we find a constraint on the various unspecified functions:

$$\chi_{1,r}\{\chi_{1,r} + n_1[F_0(\chi_1) - \psi_0]kr\} \times \{n_1[\chi_0 - k(F_0(\chi_1) + \psi_0)] - \chi_{1,r}/r\} = 0. \tag{29}$$

We see that there are two basic classes of solutions depending on whether

$$\chi_{1,r} + n_1(F_0 - \psi_0)kr = 0 \tag{30a}$$

or

$$n_1\{\chi_0 - k[F_0(\chi) + \psi_0]\} - \chi_{1,r}/r = 0. \tag{30b}$$

In the first case we readily find from (26)

$$E_0(\chi_1)^2 - 1 = [F_0(\chi_1) - \psi_0]^2, \tag{31a}$$

while in the second case

$$(1+k^2r^2)[E_0(\chi_1)^2 - 1] - [F_0(\chi_1) - \psi_0]^2 = r^2\{\chi_0 - k[F_0(\chi_1) + \psi_0]\}^2. \tag{31b}$$

In either case $E(\chi)$ and $F(\chi)$ are related by (31). We note that a consequence of (7a), (16a), (26), and (30) is that for the first class of flows

$$u_0(r) = v_0(r) = 0; \quad w_0^2(r) = E_0^2 - 1; \tag{32a}$$

while for the second class of flows

$$u_0(r) = 0; \quad v_0(r) = r\chi_0; \quad w_0^2(r) = E_0^2 - 1 - r^2\chi_0^2. \tag{32b}$$

The two classes of solutions thus correspond to the slow and fast rotational modes in the limit of zero density.[10] Since we are interested in the first class of flows we assume that (30a)–(32a) hold. We emphasize that if we had chosen a more general form than (25) so that χ, ψ, and Φ all depended on both r and ϕ, then no constraints of the form (31) would appear. It is only the special form of the assumed solutions that limits $F(\chi)$ and $G(\chi)$ up to the order we have examined the system. To the order we have solved the system $\chi_1(r)$ is arbitrary. $E_0(\chi_1)$ and $F_0(\chi_1)$ are related by (31), and $\psi_1(r,\phi)$ and $\Phi_1(r)$ are given as solutions of (27) and (28).

We next turn to the second order calculation. We now have

$$\psi = \psi_0(r) + \epsilon\psi_1(r,\phi) + \epsilon^2\psi_2(r,\phi), \tag{33a}$$

$$\chi = \chi_0 + \epsilon\chi_1(r) + \epsilon^2\chi_2(r,\phi), \tag{33b}$$

$$\Phi = \epsilon\Phi_1(r) + \epsilon^2\Phi_2(r,\phi), \tag{33c}$$

$$E(\chi) = E_0(\chi_1) + \epsilon[E_0'(\chi_1)\chi_2 + E_1(\chi_1)], \tag{33d}$$

$$F(\chi) = F_0(\chi_1) + \epsilon[F_0'(\chi_1)\chi_2 + F_1(\chi_1)]. \tag{33e}$$

From (17) we find an expression for n_2, which we may simplify by eliminating $E_0'(\chi_1)$, which we obtain from the derivative with respect to r of (31a), and we obtain

$$\begin{aligned} n_2k^2r^2[F_0(\chi_1) - \psi_0]^2 &= \chi_2[kr\chi_{1,r}F_0'(\chi_1) + (1+k^2r^2)\chi_0] \\ &\quad + kr\chi_{2,r}[\psi_0 - F_0(\chi_1)] \\ &\quad + n_1\{(1+k^2r^2)E_0(\chi_1)[E_1(\chi_1) + \Phi_1] \\ &\quad - [F_0(\chi_1) - \psi_0][F_1(\chi_1) - \psi_1]\}. \end{aligned} \tag{34}$$

We now find easily that

$$\Delta\Phi_2 = n_2E_0(\chi_1) + n_1[\Phi_1 + E_0'(\chi_1)\chi_2 + E_1(\chi_1)], \tag{35}$$

$$\begin{aligned} \Delta^*\psi_2 &= \{n_1[\psi_1 - F_0'(\chi_1)\chi_2 - F_1(\chi_1)] \\ &\quad + \{n_2[\psi_0 - F_0(\chi_1)]\}\}/(1+r^2k^2) \\ &\quad - 2k\chi_2/(1+k^2r^2)^2, \end{aligned} \tag{36}$$

$$\begin{aligned} \chi_{2,\phi\phi} + \{\chi_0/[k(\psi_0 - F_0)]\}^2\chi_2 &= -\frac{n_1\chi_0(\psi_1 - F_1)}{k^2(\psi_0 - F_0)} + \frac{n_1r\psi_{1,r}}{k(1+k^2r^2)} + \frac{r^2\chi_1n_1}{1+k^2r^2} \\ &\quad + \frac{n_1E_0}{k(\psi_0 - F_0)}\left(\frac{\chi_0(\Phi_1 + E_1)}{k(\psi_0 - F_0)} - r\Phi_{1,r}\right). \end{aligned} \tag{37}$$

On examination of systems (27), (28), (35), (36), and (37) we see that two distinct basic types of solutions are possible. We may have solutions of (27) [and possibly (28)] in which the boundary conditions require that $\psi_1(r,\phi)$ [and possibly $\Phi_1(r,\phi)$] is nontrivially dependent on ϕ, or we may have solutions in which all first order quantities depend on r only. In the first case ψ_1 is a function of r and ϕ but χ_1 is a function of r only and χ_2 is a function of both r and ϕ. We identify such solutions as externally driven helical wiggler solutions. In the second case the helical effects appear in all variables only in second order. We identify this solution as a self-excited wiggler-free solution. We treat these two solutions separately in the two next sections.

It is of some interest to decide whether or not ultrarelativistic flows for which $|w| \gg 1$ are included in our approximate treatment. It is not immediately obvious that such flows are included in an expansion in a small parameter, as a perturbation expansion assumes that all coefficients are $O(1)$ and not large. In fact, ultrarelativistic flows are included in our expansion. In ultrarelativistic flows χ, ψ, Φ, E, and F are all large and of the same order of magnitude. If we examine the system (12′), (3a″), and (20), with the definition of n, (17), then it is easy to see that all terms are of the same order of magnitude in the ultrarelativistic parameter. Thus, the expansion in ϵ does not interfere with the ultrarelativistic character of the system. In fact we may easily obtain such flows merely by assuming that $\chi_0 \gg 1$.

IV. THE EXTERNALLY DRIVEN SYSTEM

We now assume that either externally applied helically symmetric currents or that helically symmetric boundary conditions require that $\psi_1(r,\phi)$ have nontrivial ϕ dependence. For simplicity we assume that Φ_1 is a function of r only, although we could easily include ϕ dependence in Φ_1 if it were appropriate. We may represent the solution of (27) in the form

$$\psi_1 = \psi_1^r(r) + \sum_{M=1}^{\infty} (\alpha_M \cos M\phi + \beta_M \sin M\phi) r [I_M(Mkr)]_{,r}, \quad (38)$$

where $I_M(z)$ is the usual Bessel function of imaginary argument and $\psi_1^r(r)$ satisfies

$$(1/r)[r\psi_1^r(r)_{,r}/(1+k^2r^2)]_{,r} = n_1(\psi_0 - F_0)/(1+k^2r^2) - 2k\chi_1/(1+k^2r^2)^2. \quad (39)$$

With the use of (30a) to eliminate n_1 we may integrate (39) and if we assume, without loss of generality $\chi_1(0) = 0$, then

$$\chi_1(r) = kr\psi_{1,r}^r. \quad (40)$$

Equations (40), (8a), and (11) show that

$$\mathbf{B}_1^r = \hat{\phi}\chi_1/(kr), \quad (41)$$

where $\mathbf{B}_1^r$ is the ϕ independent part of $\mathbf{B}$ correct to order ϵ. Clearly $\mathbf{B}_1^r$ is the self-magnetic fields of a beam with no perpendicular momentum. The current to first order is also determined by χ_1 and is

$$-\hat{z}n_1w_0 = \hat{z}\chi_{1,r}/[kr(1+k^2r^2)]. \quad (42)$$

In order to characterize the first order flow we must specify $\chi_1(r)$ [or equivalently, by (40), $\psi_1(r)$] and $n_1(r)$ [or equivalently, by (30a), $F_0(\chi_1)$]. The electrostatic potential is the solution of (28), where $E_0(\chi_1)$ is given by (31a).

In order to compare our analysis with other treatments we simplify (38) to

$$\psi_1 = \psi_1^r(r) + Ar[I_M(Mkr)]_{,r} \sin M\phi. \quad (43)$$

Free-electron lasers have been considered usually for $M=1$ and recently also for $M=2$.[9] We can treat the problem easily for arbitrary M. We now turn to the evaluation of second order quantities and we express $\chi_2(r,\phi)$ in a form similar to (43),

$$\chi_2(r,\phi) = \chi_2^r(r) + \chi_2^\phi(r)\sin M\phi, \quad (44)$$

and from (37) we find

$$\chi_2^\phi(r) = A[n_1(r)/k]\{\chi_0[rI_M(Mkr)]_{,r}/[k(\psi_0 - F_0)] - M^2 I_M(Mkr)\}/\{M^2 - \chi_0^2/[k(\psi_0 - F_0)]^2\} \quad (45)$$

and

$$\chi_0^2\chi_2^r = n_1[-\chi_0(\psi_0 - F_0)(\psi_1^r - F_1) + r^2\chi_1(\psi_0 - F_0)^2/(1+k^2r^2) + E_0\chi_0(\Phi_1 + E_1) - rE_0\Phi_{1,r}k(\psi_0 - F_0)]. \quad (46)$$

We may consider $\chi_2^r(r)$ as arbitrary and determining, from (46), $E_1(\chi_1)$ or $F_1(\chi_1)$.

We identify immediately the appearance of a resonance in (45) whenever the denominator vanishes or whenever $\chi_0/[k(\psi_0 - F_0)]$ take on an integer multiple of M. For the case $M=1$ steady-state solutions where $\chi_0/[k(\psi_0 - F_0)] = -\chi_0/(kW_0)$ is close to one are known to be favorable for the operation of certain free-electron lasers. In general the perturbation solution of the original system fails very near a resonance. Solutions might exist if the profiles are chosen such that at values of r for which the denominator in (45) vanishes it is also true that the numerator vanishes. It is plausible that provided suitable constaints are satisfied that the original system (12′), (3a″), and (20) has solutions even if isolated resonances occur. In the next section we examine a flow that is everywhere resonant; for the remainder of this section we assume that no resonance occurs in the domain of interest or that $\chi_0/[k(\psi_0 - F_0)]$ is bounded away from integer values. Once we have determined χ_2, we may find ψ_2 and Φ_2 from (35) and (36).

We may express the reduced momentum explicitly by (7a), (16a), and (16b)

$$u = -\epsilon\chi_{2,\phi}/(rn_1) + O(\epsilon^2), \quad (47a)$$

$$v = \epsilon\{-\chi_0\chi_2/[n_1kr(\psi_0 - F_0)] - E_0(E_1 + \Phi_1)/[kr(\psi_0 - F_0)] + (F_1 - \psi_1)/(kr)\} + O(\epsilon^2), \quad (47b)$$

$$w = F_0 - \psi_0 + \epsilon\{\chi_2\{F_0' + \chi_0/[n_1(\psi_0 - F_0)]\} - E_0(E_1 + \Phi_1)/(\psi_0 - F_0)\} + O(\epsilon^2). \quad (47c)$$

The reduced momenta are sums of terms which are functions of r alone plus terms which are functions of both r and ϕ. The ϕ-dependent terms are proportional to the amplitudes of the applied helically symmetric magnetic field. The terms purely dependent on r contain the effects of the self-fields of the beam and correspond to the usual Brillouin flow of a cylindrical beam.

For the remainder of this section we consider the particularly simple zero density limit of the flow. Although the formulas (47) for the reduced particle momentum are quite simple, it is perhaps useful to make contact with other treatments of this problem. From (47c) we see that the leading order component of the z component of the momentum is $F_0 - \psi_0$, which we hold fixed as we let $n_1 \to 0$. From (30a) we see that χ_1 is $O(n_1)$, and by (42) we also take $\psi_1^r = O(n_1)$. Thus, consistent with n_1 small we take $\Phi_1 = O(n_1)$ and $F_1 = E_1 = 0$. Thus, in the limit as $n_1 \to 0$ and with $\psi_1^r = 0$ and ψ_1 given by (43) we find

$$u_1 = -\frac{MA}{kr}\left(\frac{\chi_0 r}{k(\psi_0 - F_0)}[I_M(Mkr)]_{,r} - M^2 I_M(MKr)\right)\cos M\phi\left[M^2 - \left(\frac{\chi_0}{k(\psi_0 - F_0)}\right)^2\right]^{-1}, \tag{48a}$$

$$v_1 = -\frac{AM^2}{kr}\left(r[I_M(MKr)]_{,r} - \frac{\chi_0}{k(\psi_0 - F_0)} I_M(MKr)\right)\sin M\phi \times\left[M^2 - \left(\frac{\chi_0}{k(\psi_0 - F_0)}\right)^2\right]^{-1}, \tag{48b}$$

$$w_1 = \frac{A}{k(\psi_0 - F_0)}\left(\frac{F_{0,r}}{kr} + \chi_0\right)\left(\frac{\chi_0 r}{k(\psi_0 - F_0)}[I_M(Mkr)]_{,r} - M^2 I_M(Mkr)\right)\sin M\phi\left[M^2 - \left(\frac{\chi_0}{k(\psi_0 - F_0)}\right)^2\right]^{-1}. \tag{48c}$$

Near the axis, we can expand the Bessel function in terms of the small argument Mkr and find the momentum to lowest order in r

$$u_1 = u_\perp \cos M\phi, \tag{49a}$$

$$v_1 = -u_\perp \sin M\phi, \tag{49b}$$

$$w_1 = -[u_\perp r/M(\psi_0 - F_0)](\chi_0 + F_{0,r}/kr)\sin M\phi, \tag{49c}$$

where

$$u_\perp = \frac{MAk(\psi_0 - F_0)}{(M-1)!}\left(\frac{M}{2}\right)^M \frac{(kr)^{M-1}}{[\chi_0 + Mk(\psi_0 - F_0)]}. \tag{50}$$

The perpendicular momentum is large when the resonant denominator

$$\chi_0 + Mk(\psi_0 - F_0) = B_{z0} - Mkw_0 \tag{51}$$

is small. For the usual wiggler, when $M = 1$, this reduces to the well-known cyclotron resonance.[2] On axis the momentum is

$$u_1 = u_{\perp 0} \cos\phi, \tag{52a}$$

$$v_1 = -u_{\perp 0} \sin\phi, \tag{52b}$$

$$w_1 = 0, \tag{52c}$$

where

$$u_{\perp 0} = \frac{Ak(\psi_0 - F_0)}{2[\chi_0 + k(\psi_0 - F_0)]} = -\frac{Akw_0}{2(B_{z0} - kw_0)} \tag{53}$$

as in previous one-dimensional studies. Most recently there was some interest in the use of a quadrupole wiggler.[9] On the axis the momentum is zero, and near the axis it is

$$u_1 = u_{\perp q} \cos 2\phi, \tag{54a}$$

$$v_1 = -u_{\perp q} \sin 2\phi, \tag{54b}$$

$$w_1 = 0, \tag{54c}$$

where

$$u_{\perp q} = \frac{2Ak(\psi_0 - F_0)kr}{[\chi_0 + 2k(\psi_0 - F_0)]} = -\frac{2Ak^2 w_0 r}{(B_0 - 2kw_0)}. \tag{55}$$

One can see from (49)–(55) that in the appropriate limit our formulation includes many standard results. However, our formulation easily encompasses self-field effects, nonlinear effects, and avoids the paraxial approximation. Moreover, this approximate solution, which describes a helical perturbation to a cylindrical beam is likely to be closer to high-density beam experiments than previous descriptions which assumed axicentered trajectories for all the beam particles. The nonlinear partial differential equations (12′), (3a″), and (20) should allow relatively easy numerical computation of complex helically symmetric, cold, relativistic flows.

V. THE SELF-EXCITED SYSTEM

We next examine the more intricate case when no external helically symmetric fields are applied in first order. We return to the general representations of the approximations in Sec. III. We now have all first order quantities χ_1, Φ_1, and ψ_1 as functions of r only. In order that we be able to generate a solution of the second order equations (35)–(37) which has nontrivial dependence on ϕ in $\chi_2(r,\phi)$ we must require that the flow be in resonance for all values of r or that for some integer N

$$\chi_0 = Nk\,[\psi_0(r) - F_0(\chi_1(r))], \tag{56}$$

so that from (30a) and (31a) we find

$$\chi_{1,r} = r\chi_0 n_1(r)/N \tag{57}$$

and

$$E_0^2 = 1 + \chi_0^2/(N^2k^2). \tag{58}$$

From (32a) we conclude that $w_0(r)$ is independent of r and $w_0^2 = \chi_0^2/(Nk)^2$. A further consequence of the constancy of E_0 is that $E(\chi) = E_0 + \epsilon E_1(\chi_1) + O(\epsilon^2)$ and w_1 is a function of r only given by

$$w_1(r) = E_0(E_1(\chi_1) + \Phi_1(r))/w_0.$$

By a possible redefinition of the constant χ_0, we may assume $\chi_1(0) = 0$, and just as before we may integrate (27) for ψ_1 to obtain now, cf. (40),

$$\chi_1(r) = kr\psi_{1,r}, \tag{59}$$

while a first integral of (28) for Φ_1 yields

$$\chi_1(r) = [\chi_0/(E_0N)]r\Phi_{1,r}. \tag{60}$$

On comparing (59) and (60), integrating, and selecting the irrelevant constants $\psi_1(0)$ and $\Phi_1(0)$ to be zero, we find

$$\chi_0\Phi_1(r) = kNE_0\psi_1(r). \tag{61}$$

Thus, we may give $n_1(r)$ arbitrarily and then $\chi_1(r)$ is given by (57), $\psi_1(r)$ by (59), $\Phi_1(r)$ by (61), $F_0(\chi_1(r))$ by (56), and the constant E_0 by (58).

One may notice that Eqs. (57)–(61) describe the relations among the self-electromagnetic fields of a monoenergetic beam with zero perpendicular momenta. In the self-excited system we are not free to specify both χ_1 and E_0 (or equivalently n_1 and w_0). The condition for the existence of a helical solution requires that E_0 and w_0 be r independent and be determined by χ_0, N, and k. Thus, we are free to specify either $\chi_1(r)$ or $n_1(r)$ and the other is given by (57). The lowest-order nonzero electric field $E_{r1}(r)$ is related to $B_{\theta 1}(r)$ by (61) and

$$E_{r1}(r) = -E_0 B_{\theta 1}(r)/w_0.$$

We now start an examination of the second order approximation system (35)–(37). The solution of (37) is

$$\chi_2(r,\phi) = \chi_2^r(r) + \chi_2^\phi(r)\sin N\phi, \tag{62}$$

where

$$\chi_2^r(r) = n_1(r)[E_0(E_1+\Phi_1)/\chi_0 - \chi_1/\chi_0^2 - (\psi_1 - F_1)/(kN)]. \tag{63}$$

The profile $\chi_2^\phi(r)$ is not yet specified, and the major task of this section is to complete the specification of the helical components of all second order quantities. In preparation for this task we note that we may simplify (34) with (56)–(62) to obtain

$$n_2 r^2 \chi_0^2/N^2 = \chi_0 \chi_2^\phi(r)\sin N\phi + r\chi_{2,r}\chi_0/N - \chi_1 n_1/\chi_0 - n_1 k^2 r^2 E_0(E_1+\Phi_1). \tag{64}$$

On differentiation of (56) with respect to r, together with (22b) and (57), we find

$$F_0'(\chi_1) = -kN/n_1(r), \tag{65}$$

so that differentiation of (63) with respect to r yields

$$\chi_{2,r}^r/n_1 = (rn_1/N)[E_1'E_0 + \chi_0(F_1' + F_0''\chi_2^r)/(kN)] + \chi_1 N/(r\chi_0^2) - rn_1/(\chi_0 N). \tag{66}$$

When we express $\Phi_2(r,\phi)$ and $\psi_2(r,\phi)$ in forms similar to (62),

$$\Phi_2(r,\phi) = \Phi_2^r(r) + \Phi_2^\phi(r)\sin N\Phi,$$
$$\psi_2(r,\phi) = \psi_2^r(r) + \psi_2^\phi(r)\sin N\Phi, \tag{67}$$

we obtain from (35) and (36)

$$(r\Phi_{2,r}^\phi)_{,r}/r - N^2(1+k^2r^2)\Phi_2^\phi/r^2 = E_0(Nr\chi_{2,r}^\phi + N^2\chi_2^\phi)/(r^2\chi_0) \tag{68}$$

and

$$[r\psi_{2,r}^\phi/(1+k^2r^2)]_{,r}/r - N^2\psi_2^\phi/r^2 = [\chi_2^\phi/(1+k^2r^2)]_{,r}/(kr) + N\chi_2^\phi(kr^2). \tag{69}$$

We could also obtain explicit relations for ψ_2^r and Φ_2^r, but we do not need them to complete the specification of the helical components.

Equations (68) and (69) are two relations for the three known quantities χ_2^ϕ, ψ_2^ϕ, and Φ_2^ϕ. In order to obtain the necessary additional equation we must return to (20) and expand to one higher order. In fifth order, the equation comparable to (37) is

$$\chi_{3,\phi\phi} + N^2\chi_3 = f(\chi_1,\psi_1,\Phi_1,\chi_2,\psi_2,\Phi_2,r), \tag{70}$$

for some given function $f(\chi_1,\psi_1,\Phi_1,\chi_2,\psi_2,\Phi_2,r)$. In general (70) will have a solution which is periodic in ϕ only if there are no terms on the right-hand side proportional to $\cos N\phi$ or $\sin N\phi$. In the usual terminology we must eliminate the secular terms in (70). The condition that the coefficient of $\sin N\phi$ vanish, which is the requirement that there be no secular terms, yields the final relation. Then, $\chi_3(r,\phi)$ is determined up to the addition of an arbitrary linear combination $a(r)\cos N\phi + b(r)\sin N\phi$. The unknown functions $a(r)$ and $b(r)$ will be determined in sixth order to eliminate secular terms. Thus, it is clear that this procedure can proceed order by order.

We now return to (20) and we expand to fifth order; in the process we drop all third order potentials, which are given by the left-hand side of (70), and we drop any terms which are clearly not proportional to $\sin N\phi$. Rather than use the symbol $=$, we manipulate the equation with the symbol $\equiv$, to denote equality modulo terms containing third order potentials, or terms *not* proportional to $\cos N\phi$ or $\sin N\phi$, or terms of higher order than ϵ^5. We may simplify (20) considerably if we note, for instance that

$$(\chi_{,\phi})^2 = \epsilon^4(\chi_2^\phi)^2(1+\cos 2N\phi)/2 + O(\epsilon^5),$$

so that we may effectively drop all such terms in (20), and (20) becomes

$$-2\chi_{1,r}\chi_{2,r}^r N^2\chi_2^\phi \sin N\phi \equiv -r^2n^2[\Phi + E(\chi)]\chi_{,r}\Phi_{,r} - r\chi_{,r}[krn(\psi - F) - \chi_{,r}]^2/(1+k^2r^2)^2 + r^2n\chi_{,r}[n(\psi-F)\psi_{,r} + \chi\chi_{,r}/(1+k^2r^2)]. \tag{71}$$

It is easy to conclude

$$-r^2n^2(\Phi+E)\chi_{,r}\Phi_{,r} \equiv -\epsilon^5[3rn_1^2E_0^2N(\chi_1/\chi_0)\chi_{2,r}^\phi + r^3n_1^3E_0\chi_0\Phi_{2,r}^\phi/N + 2n_1^2E_0^2N^2\chi_1\chi_2^\phi/\chi_0]\sin N\phi \tag{72}$$

and

$$-r\chi_{,r}[krn(\psi-F) - \chi k_r]^2/(1+k^2r^2)^2 \equiv \epsilon^5 2n_1^2\chi_1 N(\chi_2^\phi/\chi_0)\sin N\phi. \tag{73}$$

We define the remaining term in (71)

$$T = n\chi_{,r}[n(\psi-F)\psi_{,r} + \chi\chi_{,r}], \tag{74}$$

and after some simplification

$$T \equiv -\epsilon^5 n_3 n_1^2\chi_0^3 r^2/N^2 + \epsilon^5 n_2\chi_0^2 n_1^2[-kr^2(\psi_1 - F_1 + kN\chi_2/n_1)/N + \chi_1/(k^2N^2)] - \epsilon^5 kr^2\chi_0^2 n_1^3(\psi_2 - F_0''\chi_2^r\chi_2 - F_1'\chi_2)/N + \epsilon^5\chi_1\chi_0\chi_2 n_1^2 + \epsilon^5\chi_0^2 n_1^3[r/(kN^2)]\psi_{2,r} + \epsilon^5 r\chi_0\chi_1 n_1^2\chi_{2,r}/N + \epsilon^5 r^2\chi_0^2(n_1^3/N^2)\chi_2 + \epsilon^5 n_1(\chi_{2,r} + n_2 r\chi_0/N)(1+k^2r^2)(N/r)(-\chi_0\chi_2^\phi \sin N\phi + \chi_1 n_1 E_0^2/\chi_0). \tag{75}$$

In order to proceed we need an explicit form for n_3. We return to the definition (17) and we find, after some calculations

$$(r^2\chi_0^2/N^2)n_3^\phi \equiv -\chi_2^\phi[2k^2N^2E_0(E_1+\phi_1)/\chi_0 + N^2\chi_1(2+k^2r^2)/(\chi_0^2r^2) + n_1/\chi_0 + n_1k^2r^2E_0E_1'] - n_1[(1+k^2r^2)E_0\Phi_2 - \chi_0\psi_2/(Nk)] - [N/(r\chi_0)]\chi_{2,r}^\phi[\chi_1/\chi_0 + k^2r^2E_0(E_1+\phi_1)]. \quad (76)$$

We must next insert the expression (76) into (75) and then collect terms in (71). We find the remaining relation needed to close the system (68) and (69)

$$\alpha\psi_{2,r}^\phi + \beta\Phi_{2,r}^\phi + \gamma\psi_2^\phi + \delta\Phi_2^\phi + \zeta\chi_2^\phi = 0, \quad (77)$$

where

$$\alpha = n_1r\chi_0^2/[kN^2(1+k^2r^2)], \quad (78a)$$

$$\beta = -n_1r\chi_0E_0/N, \quad (78b)$$

$$\gamma = -n_1\chi_0^2(kN), \quad (78c)$$

$$\delta = n_1\chi_0E_0, \quad (78d)$$

$$\zeta = 2\chi_1N(N+1)/(\chi_0r^2) + [2k^2N^2E_0(E_1+\Phi_1) + n_1(1+k^2r^2E_0^2)/(1+k^2r^2)]. \quad (78e)$$

The system of equations is, then, (68), (69), and (77), together with the definitions (78). In this formulation the constants χ_0, N, and k are given, together with the essentially arbitrary functions $n_1(r)$ and $E_1(\chi_1)$. The functions $\chi_1(r)$, $\psi_1(r)$, and $\Phi_1(r)$ are given by (57), (59), and (60). The complete specification of the second order stream function requires, see (63), the additional arbitrary function $F_1(\chi_1)$, or equivalently we may consider $\chi_2^z(r)$ as an arbitrary function. We could easily express $\psi_2^z(r)$ and $\Phi_2^z(r)$ in terms of this data as well. We now return to the basic system (68), (69), and (77). We could use (77) to eliminate $\chi_2^\phi(r)$ from (68) and (69) and we would have two linear second order ordinary differential equations for $\psi_2^\phi(r)$ and $\Phi_2^\phi(r)$. The second derivatives of these functions appear in both equations. These two equations are possibly singular at points at which one cannot solve explicitly for the second derivatives. It is easy to conclude that the condition for this new resonance and singularity is

$$\chi_1(N+1)/(\chi_0r^2) + k^2NE_0(E_1+\Phi_1) = 0. \quad (79)$$

Thus, if we give the profiles $n_1(r)$, $E_1(\chi_1)$, and constants χ_0, N, and k we must ensure that (79) holds at no points in the range of values of r of interest. If (79) holds identically the system (68), (69), and (77) might be well behaved, but considerable further analysis would be needed to decide. Provided (79) fails the system (68), (69), and (77) is a fourth-order system. When (79) holds identically, the system is at most third order, although the order could be lower.

It is of some interest to examine whether fully self-excited solutions of the system are possible. By a fully self-excited solution, we mean one in which the helical magnetic and electrostatic fields tend to zero as $r\to\infty$. After our reduction of the system to two second order differential equations [assuming (79) never holds] we are left with an eigenvalue problem in which we look for nontrivial solutions of our system that are regular at the origin and that vanish at infinity. It is quite plausible that such a system would have nontrivial solutions for some parameter values, but it is by no means assured that nontrivial solutions exist. We can show fairly easily, however, that such solutions exist. We pick $\chi_2^\phi(r)$ essentially arbitrarily, solve (68) and (69) for $\psi_2^\phi(r)$ and $\Phi_2^\phi(r)$ and then we use (77) to determine the function $E_1(\chi_1)$. Thus, we show for some $E_1(\chi_1)$ there are indeed fully self-excited steady flows. We may rewrite (77) as

$$2k^2N^2E_0(E_1+\Phi_1) = -2\chi_1N(N+1)(\chi_0r^2) - n_1(1+k^2r^2E_0^2)/(1+k^2r^2) - (\alpha\psi_{2,r}^\phi + \beta\Phi_{2,r}^\phi + \gamma\psi_2^\phi + \delta\Phi_2^\phi)/\chi_2^\phi. \quad (77')$$

In order that (77') define $E_1(\chi_1)$ acceptably, it is sufficient that $\chi_2^\phi(r)$ never vanish. It would also be desirable that for small r the quantities $r\psi_{2,r}^\phi/\chi_2^\phi, \psi_2^\phi/\chi_2^\phi, r\Phi_{2,r}^\phi/\chi_2^\phi$, and Φ_2^ϕ/χ_2^ϕ all be regular. If we cannot satisfy this constraint then we must take $n_1(r)$ to vanish to some high order near $r=0$, which gives a hollow density profile. For large values of r we expect a reasonable solution whatever the functions may be, since we may take $n_1(r)$ as vanishing rapidly as $r\to\infty$. We now look to the solvability of (68) and (69) with the right-hand sides as given.

The solution of

$$(r\Phi')'/r - (N^2/r^2 + N^2k^2)\Phi = S(r), \quad (80)$$

which is regular at the origin and which vanishes at infinity is

$$\Phi = l\left(I_N(Nkr)\int_r^\infty S(r')r'\,dr'\,K_N(Nkr') - K_N(Nkr)\int_0^r S(r')r'\,dr' I_N(Nkr')\right), \quad (81)$$

for some constant l. It is easy to show that if $S(r)\sim r^M$ for r small then $\Phi\sim r^{M+2}$. Thus, turning to (68) we see that if $\chi_2^\phi\sim r^P$ then $\Phi_2^\phi\sim r^P$ also. A similar argument applied to (69) shows that $\psi_2^\phi\sim r^P$. Hence (77') is well defined as $r\to 0$. We could easily examine the behavior of (77') for r large by similar methods, but we insure that (77') is well behaved for r large merely by selecting a profile $n_1(r)$ which vanishes rapidly as $r\to\infty$. Thus, we have exhibited fully self-excited solutions of (68), (69), and (77) for appropriate $n_1(r)$ and $E_1(\chi_1)$.

We might also consider the low-density limit, exactly as we did in the last section. We must, however, consider two possibilities for a low-density limit. From (57), (59), and (60) we see that for $n_1(r)$ small χ_1, Φ_1, and ψ_1 are all small of the order of density. Now, if we take $E_1(\chi_1)$ also to be of order of the density, then a reasonable scaling would have χ_2^ϕ, ψ_2^ϕ, and Φ_2^ϕ also of order of the density, and then *no* simplification of the system occurs as in each of the equations (68), (69), and (77) all the terms are of the same order of magnitude. Thus, with $E_1(\chi_1)$ of order n_1 fully self-excited solutions are possible. Alternatively we might take $E_1(\chi_1)$ of order of one in the low-density limit. In this case (77) shows that ψ_2^ϕ is of order of the density, so that from (68) and (69) we see that ψ_2^ϕ and Φ_2^ϕ are vacuum fields. Either magnetic or electrostatic fields may excite helical flow and we consider magnetic excitation

$$\psi_2^\phi = ArI_N'(Nkr), \quad (82a)$$

$$\Phi_2^\phi = 0 \tag{82b}$$

so that

$$2k^3N^3E_0E_1\chi_2^\phi = -n_1\chi_0^2 Ar[N^2I_N(Nkr)/(Nkr) - I'_N(Nkr)]. \tag{83}$$

In this case external magnetic field wiggler sources are present but both the helical fields and the helically perturbed stream function are of second order. We require smaller externally applied wiggler fields than for the flows in Sec. IV. The expression (83) is to be contrasted with (45).

It is often useful to have general explicit expressions for the reduced momenta. From (7a), (16a), and (16b) we find

$$u = \epsilon(\chi_2^\phi \cos N\phi)N/(n_1r) + O(\epsilon^2), \tag{84a}$$

$$v = -\epsilon(\chi_2^\phi \sin N\phi - \chi_1 n_1/\chi_0^2)N/(n_1r) + O(\epsilon^2), \tag{84b}$$

$$w = -[\chi_0/(Nk)]\{1 + \epsilon[(N^2k^2/\chi_0^2)E_0(E_1 + \Phi_1)]\} + O(\epsilon^2). \tag{84c}$$

Note that no helical effects appear in the first order axial reduced momentum.

Although it is an open question of some interest we cannot be sure whether exact, nonlinear solutions of our original system exist which are of the self-excited type. Clearly the perturbation expansion described here can be extended to all orders. It is quite plausible that the nonlinear equation (20) possesses periodic solutions provided appropriate constraints on the unknown function $F(\chi)$ and $E(\chi)$ are satisfied. We must, unfortunately, leave this vital question open.

VI. CONCLUSIONS

We have presented a general cold fluid theory of helical relativistic non-neutral steady state flow which includes self-fields and radial dependence effects. The theory was applied to externally driven systems (wiggler fields present) and to self-excited systems ("wiggler-free" flows). An approximate set of equations was derived under the assumption of small perpendicular momenta. The approximate steady states are especially suitable for high-density thick beams where previous models are inadequate. We have considered a general multipole magnetic field configurations and do not restrict ourselves necessarily to dipole or quadrupole fields. The steady state flows described here will be the basis of a stability analysis of free-electron laser interactions.

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